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Interacting Quantum and Classical Continuous Systems I. The Piecewise Deterministic Dynamics

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A mathematical construction of a Markov–Feller process associated with a completely positive coupling between classical and quantum systems is proposed. The example of the free classical particle on the Lobatchevski space Q interacting with the quantum system characterized by coherent states on Q is considered.

KEY WORDS: Completely positive coupling; dissipative dynamics; piecewise deterministic process.

1. INTRODUCTION

In recent years open systems started to play an important role, for example, in the field of quantum optics (ref. 18 and references therein) or in condensed-phase dynamics (ref. 19 and references therein). In such a system it is assumed that a small, quantum part interacts with a surrounding bath. The Hamiltonian of the total system takes form

$$H = H_s + H_r + V_I$$

where H_s and H_r are the Hamiltonians of the isolated quantum subsystem and the reservoir, and V_I is the coupling operator. The problem how the properties of the quantum subsystem are modified by the influence of the environment occupies the central place in the discussions. To answer this one has to introduced, the so-called, reduced density matrix, which is obtained from the total density matrix by tracing over the bath variables,

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and to derive the dynamical equation which it satisfies, the master equation. Generally the composition of a unitary automorphism with a conditional expectation leads to a complicated integro-differential equation. However, for a large class of interesting physical phenomena we can derive, using certain limiting procedures, such as the weak coupling limit, the low density limit or the singular coupling limit, an approximate Markovian master equation for the reduced density matrix.[1]

In order to obtain a particular equation a model for the bath has to be specified. A convenient choice is to assume that the reservoir consists of independent harmonic oscillators linearly coupled to the quantum subsystem. For example, in ref. 7 such a model was used for the description of properties of a quantum harmonic oscillator coupled to the radiation field. When information about the dynamics of the classical variables is unimportant for the problem, then it is convenient to treat the environment as a source of random fluctuations of the quantum Hamiltonian. Then the quantum subsystem evolves according to $H'_s = H_s + \sum_j U(q, q_i)$, where U is a potential function, q is the quantum position variable and the $\{q_i\}$ specify the possible configuration of the classical particles. For example, in such a way the model of a quantum particle in a double-well potential immersed in a classical fluid was described in ref. 13.

In all these models, however, the classical environment is, in fact, treated also as a quantum system evolving according to the Hamiltonian H_r. In another approach it is assumed that only the small part needs to be treated quantum mechanically, while the environment may be described classically, by Newton's equation in the field of the quantum subsystem. When the quantum subsystem adapts instantaneously to the dynamics of the classical bath the following computational algorithm can be applied.[22] One begins with an arrangement of classical molecules $R(t)$ at fixed initial time $t = t_0$. The adiabatic ground state $\psi_0(q)$ is found as a solution of time-independent Schrödinger equation in the interacting potential $V_I(q, R(t_0))$. The bath moves to a new configuration $R(t_1)$ according to Newton's law with the additional quantum potential

$$F_q(t) = -\langle \psi_0, \nabla_R V_I(q, R(t)) \psi_0 \rangle$$

for $t \in [t_0, t_1]$. Next the new $\psi_1(q)$ must be found for the new configuration of the classical system $R(t_1)$, and we can repeat the procedure. Thus the quantum subsystem affects the classical dynamics through the quantum force $F_q(t)$, and the quantum state depends on the classical configuration $R(t)$. Using mixed state description over short times and periodically resolving ψ into its adiabatic components we obtain a non-adiabatic generalization called "surface hopping,"[22] in which the transitions between adiabatic states are allowed. A practical algorithm that determines when a quantum state switch will occur was proposed by Tully in ref. 21. The key point in his method lies in calculating, the so-called, switching probabilities g_{kj} (depending on time) from the current quantum state $|k\rangle$ to all other states $|j\rangle$. When we know them, a uniformly distributed random number $\theta \in (0, 1)$ is then selected to determine whether a switch should occur. For example, if $k=1$, a switch to state $|2\rangle$ will occur if $\theta < g_{12}$. A switch to state $|3\rangle$ will occur if $g_{12} < \theta < g_{13}$, etc. Such methods have been successful in quite many examples.[21, 22] However, they contain two basic disadvantages. Firstly, they do not provide a consistent dynamical scheme, and, secondly, they ignore the results of the modified dynamics of the classical variables.

Recently a mathematically consistent description of the interaction between classical and quantum systems has been proposed.[3–5] From the structural and mathematical point of view the three most essential ingredients of the Blanchard and Jadczyk model are:

— tensoring of a non commutative quantum algebra of operators with a commutative classical algebra of functions,

— renouncing pure states for density matrices and replacing Schrödinger unitary dynamics by a completely positive one,

— interpreting the continuous time evolution of statistical states in terms of a piecewise-deterministic Markov process on pure states of the total system.

In this approach classical quantities become elements of the center of the total algebra. Because automorphisms of an algebra leaves its center invariant, it is necessary to use completely positive semigroups to enable the transfer of information between the classical and quantum system. Thus the evolution of the quantum object becomes dissipative and the modification of the dynamics of the classical system through the expectation value of some quantum observable appears. The fact that the classical and the quantum system must be coupled by a dissipative rather then reversible dynamics follows from no-go theorem,[12, 16] where it was shown in a general algebraic framework that the information about the measured object cannot be transmitted to values of macroscopic observable as long as the dynamics of the total system is reversible in time.

Clearly, the key point in the coupling is to construct a generator of a dynamical semigroup of the total system. Recently an example of such a generator has been introduced in ref. 14. The classical system was represented by an algebra of functions defined on some phase space (symplectic manifold M) while the quantum system was described by the von Neumann algebra of all bounded operators on a Hilbert space. The coupling operator has been built out of the following data:

(a) a self-adjoint quantum operator $\hat{P}$,

(b) a connection between the points of the spectrum $\mathrm{sp}(\hat{P})$ of $\hat{P}$ and shifts on M,

(c) a function $f: M \times \mathrm{sp}(\hat{P}) \to \mathbf{R}_+$ monitoring the strength of the coupling.

Such a generator turned out to be suitable for the rigorous discussion of the SQUID-tank model, which consists of an LC oscillatory circuit coupled via a mutual inductance to a superconducting ring. In that system the oscillatory circuit acts as an external flux source for the SQUID ring, which induces a screening current in the ring. This screening current is coupled back to the classical circuit due to the mutual inductance. It results in the modification of the equation of motion for the clasical harmonic oscillator by the expectation value of the screening current operator.[14]

When a quantum system is characterized by a semispectral measure on some homogeneous space, like in the generalized coherent state approach, then there is no particular self-adjoint operator, which could be responsible for the coupling, but all quantum states can affect the classical variables. Such models describe a quantized particle moving on a locally compact homogeneous configuration space G/H. They are physically interesting because the nontrivial topology of the configuration space may exhibit, the so-called, topological quantum effects (see, for example refs. 10 and 11, where the particle on the circle and on the sphere was considered). Let us discuss this point more precisely. To define the interaction between the classical and quantum system one has to specify which elements of both systems are essential for the coupling. In the classical system the answer seems to be clear: points of the phase space do the job. When there is a particular self-adjoint quantum operator which is supposed to influence the behavior of the classical part, then its spectral measure, i.e., a complete family of orthogonal projectors, is the right ingredient of the coupling. In the case when all quantum states, or at least a family of states which generates the quantum algebra, can affect the classical system we have to discriminate between non-orthogonal projectors and so to replace a spectral measure by a semispectral one. The generalized coherent state are the best known example which provides such a measure. In such a case the construction of the coupling operator was presented in ref. 15.

With a given dynamical semigroup T_t we can associate a Markov–Feller process with values in the pure state space of the total system in such a way that $T_t(P_x) = \int P(t, x, dy)\, P_y$ is satisfied. Here P_x is a one-dimensional projector representing pure state x and $P(t, x, dy)$ is the transition probability function of a desired process. It consists of a mixture of deterministic motion and random jumps. In the case when a discrete classical system (a measuring apparatus) is coupled to a finite quantum system described by a matrix algebra it was shown in ref. 8 that such a process exists and moreover, contrary to the pure quantum case, that this process is unique. For the probabilistic description of this process, see ref. 6. Applying the uniqueness theorem we deduce the piecewise deterministic algorithm generating sample paths of an individual quantum system.[5] The algorithm provides a way for calculating numbers needed in real experiments and also for a natural mathematical modeling of a feedback during experiments with quantum systems. The most transparent example of the applications of such an algorithm is the computer simulation of traversal and reflection times of electrons through a one-dimensional barrier. In ref. 20 such a tunneling phenomenon was considered and the dependence of these times on the parameters of the barrier and the detectors was examined.

The main objective of the present paper is to provide the construction of a piecewise deterministic process in the case when the classical system is continuous and the quantum one infinite dimensional and to discuss its properties.

2. THE DYNAMICAL SEMIGROUP ASSOCIATED TO THE TOTAL SYSTEM

Let us briefly describe the framework for the classical-quantum coupling. At first we consider a classical system C with a finite number of degrees of freedom. Its phase space is a symplectic manifold (M, ω). The C^*-algebra $C_0(M)$ of continuous and vanishing at infinity functions represents complex observables of the system. Because it will be more convenient to consider von Neumann algebras we pass to the representation in the Hilbert space $\mathscr{H}_c = L^2(M, \mathscr{B}, \mu)$, where $\mathscr{B}$ is the Borel σ-algebra and $d\mu$ is the unique Borel measure determined by the volume form ω^n, $n = \dim M/2$. We assume that the classical algebra $\mathscr{A}_c$ equals to $C_0(M)'' = L^\infty(M, \mathscr{B}, \mu)$. Statistical states of C are then normed and positive elements of $L^1(M, \mathscr{B}, \mu)$. The time evolution of C is described by a flow on M, i.e., a mapping g: $(t, x) \to g_t(x)$ such that:

(a) g: $\mathbf{R} \times M \to M$ is smooth,

(b) for any t, g_t is a diffeomorphism of M,

(c) $t \to g_t$ is a group homomorphism.

Its generator is a complete vector field X on M. It gives an ultraweakly continuous one parameter group of automorphisms of $\mathscr{A}_c$: $f(x) \to f(g_t^{-1}x)$, $x \in M$. Its generator we denote by δ_c.

Now we come to the quantum system. Let us consider a quantum particle on a homogeneous configuration space $Q = G/K$, where G is a Lie group and K is a closed subgroup. We assume moreover that G and K are both unimodular. The quantum theory of such a system may be introduced by using the concept of generalized coherent states,[17] see also refs. 2 and 9. Let $(\pi, \mathscr{H}_q)$ be a unitary, strongly continuous and irreducible representation of G, such that for every $k \in K$ $\pi(k)\,\psi_0 = e^{ia(k)}\psi_0$ for some unit vector $\psi_0 \in \mathscr{H}_q$. It follows that for each $q \in Q$ we have a one-dimensional projector $P_q = |\pi(g)\,\psi_0\rangle\langle\psi_0\pi(g)|$, where $[g] = q$. We assume that the system of coherent states is square integrable and normalized, i.e.,

$$\int_Q d\alpha(q)\, P_q = \mathbf{1},$$

in the strong sense, where $d\alpha$ is a unique G-invariant and σ-finite Borel measure on Q. We also assume that for every $q \in Q$ the reproducing kernel $q' \to K(q, q') = \langle q, q' \rangle$ vanishes for a set of $d\alpha$-measure zero. The quantum algebra $\mathscr{A}_q$ is defined as

$$\mathscr{A}_q = \left\{ \int f(q)\, P_q\, d\alpha(q),\, f \in C_c(Q) \right\}'' = \{P_q,\, q \in Q\}''$$

and so $\mathscr{A}_q = B(\mathscr{H}_q)$, the algebra of all bounded and linear operators on $\mathscr{H}_q$. Statistical states of the quantum system are given by non-negative density matrices $\rho \in \mathscr{A}_q$ with $Tr(\rho) = 1$. The time evolution is given by $A \to e^{itH} A e^{-itH}$, where H is the operator closure of $(d\pi(h), D_G)$, $h \in \mathscr{G}$—the Lie algebra of group G, and D_G is the Gårding domain. Clearly H is a self-adjoint operator. The generator $i[H, \cdot\,]$ of the time evolution of the quantum system we denote by δ_q.

Let us now consider the joint system. For the total algebra $\mathscr{A}_T$ we take the tensor product $\mathscr{A}_T = \mathscr{A}_c \otimes \mathscr{A}_q$ as von Neumann algebras on $\tilde{\mathscr{H}} = \mathscr{H}_c \otimes \mathscr{H}_q$. The set of states is equal to

$$\mathscr{S}_T = \left\{ \tilde{\rho} \in \mathscr{A}_{T*} : \tilde{\rho}(x) \in Tr(\mathscr{H}_q)_+ \text{ a.e. and } \int_M Tr(\tilde{\rho}(x))\, d\mu(x) = 1 \right\}$$

The mean value of $\tilde{A} \in \mathscr{A}_T$ in a state $\tilde{\rho} \in \mathscr{S}_T$ is given by

$$\langle \tilde{A} \rangle_{\tilde{\rho}} = \int_M d\mu(x)\, Tr[\tilde{A}(x)\, \tilde{\rho}(x)]$$

Now let us discuss the evolution of the total system. The total generator consists of three parts: $\delta_c \otimes id$, $id \otimes \delta_q$ and a superoperator L, which describes the interaction between the classical and the quantum system.

To construct L we assume the following:

(a) To every point $x \in M$ we associate an intensity parameter $\lambda(x) > 0$. We assume that λ is continuous and such that $\sup_{x \in M} \lambda(x) < \infty$. The influence of the classical system onto the quantum one is described by a dissipative operation

$$A \to \lambda(x) \int_Q P_q A P_q \, d\alpha(q)$$

(b) To every point $q \in Q$ corresponds a shift on the phase space M. By shift we mean a homeomorphism $h_q \colon M \to M$, which leaves the measure $d\mu$ invariant. Moreover we require that the mapping

$$M \times Q \ni (x, q) \to h_q(x) \in M$$

is continuous. The shifts are responsible for the action of the quantum system on the classical one by $f \to f(h_q \cdot)$.

Theorem 2.1. Suppose $\tilde{A} \in \mathscr{A}_T$. Let

$$L(\tilde{A})(x) = \lambda(x) \int_Q d\alpha(q) \, P_q \tilde{A}(h_q x) \, P_q - \lambda(x) \, \tilde{A}(x)$$

Then L is a bounded and complete dissipation such that $L(\tilde{\mathbf{1}}) = 0$, where $\tilde{\mathbf{1}}$ is the unit in $\mathscr{A}_T$.

Proof. It follows from a more general construction presented in ref. 15, if we put the measure $v_x = \lambda(x)^{1/2} \, \delta_e$, where δ_e is the Dirac measure concentrated on the neutral element in G. ∎

The operator L acting onto the classical part of $\mathscr{A}_T$:

$$L(f \otimes \mathbf{1})(x) = \lambda(x) \int_Q d\alpha(q) \, f(h_q x) \, P_q - \lambda(x) \, f(x) \mathbf{1}$$

moves it into the whole algebra $\mathscr{A}_T$. Also the quantum part is not preserved by L.

Corollary. The operator

$$B = \delta_c \otimes id + id \otimes \delta_q + L$$

generates a dynamical semigroup T_t on the algebra $\mathscr{A}_T$.

By a dynamical semigroup we understand a weak*-continuous semigroup of contractive, completely positive and normal operators.

3. CONSTRUCTION OF THE PROCESS

In this section we associate with the dynamical semigroup T_t a Markov–Feller semigroup on $C_0(E)$ with a locally compact space E. Here C_0 denotes the Banach space of all complex, continuous functions vanishing at infinity and equipped with the sup norm. When the classical space was discrete and the quantum system was finite dimensional, then we could take for E the space of all one dimensional projectors of the joint system. Let us notice that now we do not have any normal pure states in $\mathscr{A}_T = L^\infty(M,B(\mathscr{H}_q); d\mu)$ at all. So the idea is to restrict $\mathscr{A}_T$ to a smaller C^*-algebra such that we can use the locally compact space $M \times Q$ as E.

Let $\mathscr{A}_T^0 = C_0(M) \otimes K(\mathscr{H}_q)$, where $K(\mathscr{H}_q)$ denotes all compact operators on $\mathscr{H}_q$ and the tensor product is define on the Hilbert space $L^2(M, \mathscr{B}, \mu) \otimes \mathscr{H}_q$. Clearly $\mathscr{A}_T^0$ is a C^*-algebra, although without a unit.

Theorem 3.1. $T_t\colon \mathscr{A}_T^0 \to \mathscr{A}_T^0$.

Proof. We show that there exists a dense linear subspace D in $\mathscr{A}_T^0$ such that $B\colon D \to \mathscr{A}_T^0$. At first let us consider the group $\alpha_t(A) = e^{itH} A e^{-itH}$ on $L(\mathscr{H}_q))$. It is weak*-continuous, so the function $t \to Tr\rho\alpha_t(A)$ is continuous for any $\rho \in Tr(\mathscr{H}_q)$, the trace class operators on $\mathscr{H}_q$. When we restrict α_t to $K(\mathscr{H}_q)$ we obtain a weakly continuous semigroup, hence strongly continuous as well. Thus there exists an operator norm dense subspace D_q in $K(\mathscr{H}_q)$ such that $\delta_q\colon D_q \to K(\mathscr{H}_q)$. For the classical part we take $D_c = C_c^\infty(M)$, the subspace of smooth and compactly supported functions. Then putting $D = D_c \otimes D_q$ in the algebraic sense we obtain a norm dense subspace in $\mathscr{A}_T^0$ on which $\delta_c \otimes id + id \otimes \delta_q$ is well defined and takes values in $\mathscr{A}_T^0$. Finally, we show that $L\colon \mathscr{A}_T^0 \to \mathscr{A}_T^0$. Because L is bounded, so it is enough to check this for some norm dense subspace in $\mathscr{A}_T^0$. Let $D_0 = C_c(M) \otimes Tr(\mathscr{H}_q)$ and let us take $\tilde{A} = f \otimes \rho$, where $f \in C_c(M)_+$ and $\rho \in Tr(\mathscr{H}_q)_+$. Hence for any $q \in Q$ the function

$$x \to P_q \tilde{A}(h_q x)\, P_q = f(h_q x)\, P_q \rho P_q \in C_c(M, Tr(\mathscr{H}_q))$$

and

$$\int_Q d\alpha(q) \sup_{x \in M} \|P_q \tilde{A}(h_q x)\, P_q\|_{op} \leqslant \|f\|_{\sup} \int_Q d\alpha(q)\, Tr(P_q \rho) < \infty$$

Hence the function $q \to P_q \tilde{A}(h_q x) P_q$ is Bochner integrable in $\mathscr{A}_T^0$. By linearity the same holds for all $\tilde{A} \in D_0$. Because $\lambda(x)$ is continuous and uniformly bounded so also $L: D_0 \to \mathscr{A}_T^0$. ∎

Now let us define a map

$$\mathscr{A}_T^0 \ni \tilde{A} \to F(\tilde{A})(x, q) = Tr P_q \tilde{A}(x)$$

We denote $F(\tilde{A})$ by $f_{\tilde{A}}$.

Theorem 3.2. *F is a bounded and positive operator from $\mathscr{A}_T^0$ to $C_0(E)$.*

Proof. First we show that $f_{\tilde{A}}$ is a continuous function on E. Let $P_q = |\pi(g)\psi_0\rangle\langle\psi_0\pi(g)|$ and $P_{q0} = |\pi(g_0)\psi\rangle\langle\psi_0\psi\rangle\langle\psi_0\pi(g_0)|$. Then

$$\begin{aligned}
&|f_{\tilde{A}}(x, q) - f_{\tilde{A}}(x_0, q_0)| \\
&\quad \leqslant |Tr(P_q - P_{q_0})\, \tilde{A}(x)| + |Tr P_{q_0}(\tilde{A}(x) - \tilde{A}(x_0))| \\
&\quad \leqslant |\langle \pi(g)\, \psi_0, \tilde{A}(x)(\pi(g) - \pi(g_0))\, \psi_0\rangle| \\
&\qquad + \langle (\pi(g) - \pi(g_0))\, \psi_0, \tilde{A}(x)\, \pi(g_0)\, \psi_0\rangle| + \|\tilde{A}(x) - \tilde{A}(x_0)\|_{op} \\
&\quad \leqslant 2\, \|\tilde{A}(x)\|_{op}\, \|(\pi(g) - \pi(g_0))\, \psi_0\| + \|\tilde{A}(x) - \tilde{A}(x_0)\|_{op}
\end{aligned}$$

Hence $f_{\tilde{A}}$ is continuous. Moreover

$$\sup_{x, q} |f_{\tilde{A}}(x, q)| \leqslant \sup_x \|\tilde{A}(x)\|_{op} = \|\tilde{A}\|$$

so the operator F maps $\mathscr{A}_T^0$ into $C_b(M \times Q)$, the Banach space of continuous and bounded functions, and is bounded. Let $\tilde{A} = f \otimes \rho$, where $f \in C_c(M)$ and $\rho \in Tr(\mathscr{H}_q)_+$. Then $F(\tilde{A})(x, q) = f(x)\, Tr P_q \rho$. Because $\rho = \sum_n a_n P_n$ and

$$Tr P_q P_n = |\langle q, n\rangle|^2 = |\langle \pi(g)\, \psi_0, n\rangle|^2$$

for $[g] = q$ so both functions $q \to Tr P_q P_n$ and $q \to Tr P_q \rho$ are uniformly continuous. But $Tr P_q \rho \in L^1(Q, d\alpha)$ hence $Tr P_q \rho \in C_0(Q)$ too. Let us choose a compact set $K_1 \subset M$ such that $\operatorname{supp} f \subset K_1$ and a compact set $K_2 \subset Q$ such that for every $q \in Q \backslash K_2$ there is $Tr P_q \rho < \varepsilon\, \|f\|_{\sup}$. Then

$$\sup_{(x, q) \in M \times Q \backslash K_1 \times K_2} F(\tilde{A})(x, q) \leqslant \|f\|_{\sup} \sup_{q \in Q \backslash K_2} Tr P_q \rho < \varepsilon$$

So $F(\widetilde{A}) \in C_0(M \times Q)$ and this also holds for any element from $C_c(M) \otimes Tr(\mathscr{H}_q)$. Thus the assertion follows by the continuity of the map F. ▮

We want to use the operator F to transfer the dynamical semigroup T_t onto the space $C_0(E)$. But there is the price we have to pay for replacing thge big space $\mathbf{C}P(\mathscr{H}_q)$ by a locally compact space Q. It results in the possible lost of the injectivity of the map F. Nevertheless the kernel of F has the following property.

Theorem 3.3. Ker F is a closed and T_t invariant subspace in $\mathscr{A}_T^0$. If $\widetilde{A} \geqslant 0$ and $F(\widetilde{A}) = 0$ then $\widetilde{A} = 0$.

Proof. The first statement follows from the continuity of F. The generator B of T_t can be written as $B = B_0 + L$, where $B_0 = \delta_c \otimes id + id \otimes \delta_q$ and L is a bounded operator. Let $\widetilde{A} \in \ker F$. By the Trotter product formula

$$T_t(\widetilde{A}) = \lim_{n \to \infty} (e^{(t/n) B_0} e^{(t/n) L})^n \widetilde{A}$$

and the limit exists in the norm of $\mathscr{A}_T^0$. Because $\ker F$ is closed, so it is enough to check that $\exp(tB_0) \widetilde{A} \in \ker F$ and $\exp(tL) \widetilde{A} \in \ker F$. For the classical part in B_0 we have that

$$F(e^{t\delta_c \otimes id} \widetilde{A})(x, q) = Tr P_q \widetilde{A}(g_t x) = 0$$

because $Tr P_q \widetilde{A}(y) = 0$ for any $q \in Q$ and any $y \in M$. Let us recall that $g_t x$ is the integral curve of the vector field X starting from point $x \in M$. For the quantum part

$$F(e^{tid \otimes \delta_q} \widetilde{A})(x, q) = Tr(P_q e^{itH} \widetilde{A}(x) e^{-itH}) = Tr(P_{\exp(-th) q} \widetilde{A}(x))) = 0$$

Now let us consider the dissipative part. Because L is bounded, so it is enough to show that $F(L(\widetilde{A})) = 0$. But

$$F(L(\widetilde{A}))(x, q_0) = \lambda(x) \int_Q d\alpha(q)\, Tr(P_{q_0} P_q \widetilde{A}(h_q x) P_q) - \lambda(x) F(\widetilde{A})(x, q_0)$$

$$= \lambda(x) \int_Q d\alpha(q)\, Tr(P_{q_0} P_q)\, Tr(P_q \widetilde{A}(h_q x))$$

$$= \lambda(x) \int_Q d\alpha(q)\, Tr(P_{q_0} P_q)\, F(\widetilde{A})(h_q x, q) = 0$$

To prove the second statement suppose that $\tilde{A} \geqslant 0$ and $F(\tilde{A}) = 0$. It means that, for every $x \in M$, $Tr P_q \tilde{A}(x) = 0$ for all $q \in Q$. Because $\tilde{A}(x)$ is positive and compact, so $\tilde{A}(x) = \sum a_i P_i$, where $a_i \geqslant 0$ and P_i are one-dimensional projectors. So $\sum a_i Tr(P_q P_i) = 0$. But $Tr(P_q P_i)$ can not vanish for all q, since $\int d\alpha(q)\, Tr(P_q P_i) = 1$. Hence $a_i = 0$ for all i. ∎

Now let us define a vector field Y on E by

$$(Yf)(x, q) = \frac{d}{dt} f(g_t x, \exp(-th)\, q)_{t=0}$$

for all $f \in C_c^1(E)$. Let ν be a function ν: $E \times \mathscr{B}(E) \to [0, 1]$ given by

$$\nu(x, q; dx', dq') = \delta(x' - h_{q'} x)\, Tr(P_q P_{q'})\, d\mu(x')\, d\alpha(q')$$

where $\delta(x' - x)\, d\mu(x')$ denotes the Dirac measure concentrated in point $x \in M$. It is clear that ν is a transition kernel on E, that is:

(a) $\forall (x, q) \in E$ $\nu(x, q; \cdot)$ is a probabilistic measure on the Borel σ-algebra $\mathscr{B}(E)$,

(b) $\forall \mathscr{E} \in \mathscr{B}(E)$ the function $(x, q) \to \nu(x, q; \mathscr{E})$ is measurable.

Theorem 3.4. The closure of the operator $\mathscr{L}$ given by

$$\mathscr{L}(f)(x, q) = (Yf)(x, q) + \lambda(x) \int_E \nu(x, q; dx', dq')\, f(x', q') - \lambda(x)\, f(x, q)$$

for $f \in C_c^1(E)$ is a generator of a Markov–Feller semigroup P_t on $C_0(E)$. The dynamical system $(\mathscr{A}_T^0/\ker F, \hat{T}_t)$ can be imbedded into $(C_0(E), P_t)$. Here $\hat{T}_t$: $\mathscr{A}_T^0/\ker F \to \mathscr{A}_T^0/\ker F$ denotes the quotient semigroup.

Proof. To prove the first statement it is enough to show that for any $f \in C_0(E)$ also

$$(\nu f)(x, q) = \int_E \nu(x, q; dx', dq')\, f(x', q') \in C_0(E)$$

At first we check that it is a continuous function. Because ν is bounded in the sup-norm, so we may assume that $f \in C_c(E)$. Let $q_n \to q$ and $x_n \to x$. Then

$$\lim_{n \to \infty} \int_E \nu(x_n, q_n; dx', dq')\, f(x', q')$$

$$= \lim_{n \to \infty} \int_K d\alpha(q')\, Tr(P_{q_n} P_{q'})\, f(h_{q'} x_n, q')$$

where $K = pr_2(\text{supp}\, f)$ and pr_2 is the projection from $M \times Q$ onto Q. But for every n there is

$$Tr(P_{q_n} P_{q'})\, |f(h_{q'} x_n, q')| \leqslant \|f\|_{\sup}$$

and for any q'

$$\lim_{n \to \infty} Tr(P_{q_n} P_{q'})\, f(h_{q'} x_n, q') = Tr(P_q P_{q'})\, f(h_{q'} x, q')$$

so

$$\lim_{n \to \infty} \int_E \nu(x_n, q_n; dx', dq')\, f(x', q') = \int_E \nu(x, q; dx', dq')\, f(x', q')$$

Now we show that $\nu(f) \in C_0(E)$. Because for any q' the function (x, q) $Tr(P_q P_{q'})\, f(h_{q'} x, q')$ belongs to $C_0(E)$ and there is

$$\int_K d\alpha(q') \sup_{x, q} Tr(P_q P_{q'})\, |f(h_{q'} x, q')| \leqslant \alpha(K)\, \|f\|_{\sup} < \infty$$

so it is Bochner integrable and hence $\nu(f) \in C_0(E)$. It follows that $\mathscr{L}$ is a generator of a Markov–Feller semigroup on $C_0(E)$.

To prove the second statement let us define the quotient map $\hat{F}\colon \mathscr{A}^0_T/\ker F \to C_0(E)$. Then $\hat{F}$ is injective and $\hat{F}(\hat{T}_t[\tilde{A}]) = F(T_t \tilde{A})$. From the proof of Theorem 3.3 we know that

$$F(e^{t\delta_c \otimes id} \tilde{A})(x, q) = f_{\tilde{A}}(g_t x, q)$$

$$F(e^{itd \otimes \delta_q} \tilde{A})(x, q) = f_{\tilde{A}}(x, \exp(-th)\, q)$$

and

$$F(L(\tilde{A}))(x, q) = \lambda(x)\, \nu(f_{\tilde{A}})(x, q) - \lambda(x)\, f_{\tilde{A}}(x, q)$$

Hence $F(e^{tL} \tilde{A}) = e^{t\lambda(\cdot)(\nu - I)} f_{\tilde{A}}$ and so

$$F(e^{(t/n)\, B_0} e^{(t/n)\, L} \tilde{A}) = e^{(t/n)\, Y} e^{(t/n)\, \lambda(\cdot)(\nu - I)} f_{\tilde{A}}$$

Thus, by induction and the limit, we obtain that $F(T_t \tilde{A}) = P_t(F(\tilde{A}))$. Hence $\hat{F}(\hat{T}_t[\tilde{A}]) = P_t \hat{F}([\tilde{A}])$. ∎

Let us now describe the Markov process determined by the semigroup

$$P_t = s - \lim_{n \to \infty} \left(e^{(t/n)\, Y} e^{(t/n)\, \lambda(\cdot)(\nu - I)}\right)^n$$

The process generated by vector field Y is a deterministic motion on E with the transition function given by $\delta(x'-g_t x)\,\delta(q'-\exp(-th)\,q)\,d\mu(x')\,d\alpha(q')$. To calculate the transition function for the generator $\lambda(\cdot)(v-I)$ we apply the following trick. Let $\lambda=\sup_x \lambda(x)$ and let us define a new transition kernel

$$v'(x, q; dx', dq')=\left(1-\frac{\lambda(x)}{\lambda}\right)\delta(x'-x)\,\delta(q'-q)\,d\mu(x')\,d\alpha(q')$$

$$+\frac{\lambda(x)}{\lambda}\,v(x, q; dx', dq')$$

Then the jump generator may be written as

$$\lambda(x)(v-I)\,f(x, q)=\lambda\int_E [f(x', q')-f(x, q)]\,v'(x, q; dx', dq')$$

Hence the transition function for the jump process is given, for any $\Gamma\in\mathscr{B}(E)$, by

$$P_1(t, x, q; \Gamma)=e^{-\lambda t}\sum_{n=0}^{\infty}\frac{\lambda^n t^n}{n!}\,\Pi_n(x, q; \Gamma)$$

where $\Pi_0(x, q; \Gamma)=\chi_\Gamma(x, q)$ is the characteristic function of the set Γ and

$$\Pi_n(x, q; \Gamma)=\int_E v'(x, q; dx', dq')\,\Pi_{n-1}(x', q'; \Gamma)$$

for $n\geqslant 1$. It generates a step process, that is a jump process such that jump times have no limit points in any time interval $[0, T]$. It follows that the process determined by the semigroup P_t is a mixture of a deterministic motion with a step process. It generalizes the piecewise deterministic process obtained in the case of a discrete classical system and a finite dimensional quantum system.[6,8] To see this suppose that the classical system consists of m distinct points and the quantum system is described by $n\times n$ complex matrices. Hence E is a disjoint sum of m copies of the complex projective space $\mathbf{C}P^n$. Let us recall that the action of the dynamical semigroup T_t expressed in term of the process reads

$$T_t(P_x)=\int P(t, x, dy)\,P_y$$

where $P(t, x, dy)$ is the transition probability function of the process and $y \to P_y$ is the tautological map, which assigns to every point $y \in E$ the one-dimensional projector P_y. It is equivalent (Proposition 1, Section 4 in ref. 8) to the existence of a Markov semigroup U_t on the Banach space $\mathscr{M}(E)$ of all complex finite Borel measures on E, such that $\hat{U}_t = T_t$, where $\hat{U}_t$ is the quotient semigroup of U_t by $\ker \pi$ and

$$\pi(\mu) = \int_E d\mu(x)\, P_x$$

for all $\mu \in \mathscr{M}(E)$. Let us notice that in this case the map F is injective and so, by Theorem 3.4, the dynamical system $(\mathscr{A}_T, T_t)$ is imbedded into $(C(E), P_t)$. Passing to the dual semigroup and taking into account that $\mathscr{A}_T^* = (\mathscr{A}_T)_*$ and $C(E)^* = \mathscr{M}(E)$ we obtain that the equality $\hat{U}_t = T_{t*}$ holds for $U_t = P_t^*$.

4. EXAMPLE: THE COUPLING ON THE LOBATCHEVSKI SPACE

In this section we describe more precisely a Markov–Feller process associated with a dynamical semigroup for a coupled system of a non-relativistic classical particle moving freely on the Lobatchevski space and a quantum system characterized by coherent states on this space. By this example we want to demonstrate a possible influence of the quantum matter onto the classical gravitational field. It will be achieved by the change of a dynamical path of the classical particle moving freely along a geodesic curve when interacting with the quantum system. In average the classical evolution equation is perturbed by the expectation value of the quantum position operator. It fits the spirit of the semiclassical theory of gravity in which one studies the generalized Einstein's equation modified by regularized vacuum expectation value of the energy-momentum tensor of the matter-field operator.

Let us describe the classical system. We consider a classical particle moving freely on the Lobatchevski space

$$Q = \mathbf{R} \times \mathbf{R}_+ = \{(x_1, x_2) : x_2 > 0\}$$

The phase space is the cotangent space $M = T^*Q$ with the canonical symplectic form $\omega = dp_1 \wedge dx_1 + dp_2 \wedge dx_2$. It leads to the Lebesgue's measure $d\mu = dx_1\, dx_2\, dp_1\, dp_2$ on M. The time evolution is governed by a complete vector field X on M:

$$X(x, p) = \sum_{k=1}^{2} \left[f_k(x, p) \frac{\partial}{\partial x_k} + g_k(x, p) \frac{\partial}{\partial p_k} \right]$$

where $f_k(x, p) = p_k$ and $g_1(x, p) = 2p_1 p_2 / x_2$, $g_2(x, p) = (p_2^2 - p_1^2)/x_2$. For simplicity we denote the four coordinates (x_1, x_2, p_1, p_2) by (x, p). It gives the following second order differential equations for the position coordinates:

$$\ddot{x}_1 = \frac{2}{x_2} \dot{x}_1 \dot{x}_2$$

$$\ddot{x}_2 = \frac{1}{x_2} [(\dot{x}_2)^2 - (\dot{x}_1)^2]$$

In other words the classical particle moves along a geodesic curve with respect to the metric

$$g = \frac{1}{x_2^2} (dx_1 \otimes dx_1 + dx_2 \otimes dx_2)$$

Let us recall that a geodesic curve is a vertical straight line or a semicircle with the center placed in an arbitrary point on the x_1-axis.

To describe the quantum system we use the system of generalized coherent states on Q.[17] Let us recall that Q is the homogeneous space $Q = SL(2, \mathbf{R})/SO(2)$. For simplicity we take the first representation from the series $(\mathscr{H}_k, \pi_k)$, where $k = 1, 3/2, 2, \ldots$. That is

$$\mathscr{H}_q = \left\{ f \colon \|f\|^2 = \int d\mu_1(z) \, |f(z)|^2 < \infty \right\}$$

where f is a holomorphic function in the unit complex disc $|z| < 1$ and $d\mu_1 = (1/\pi) \, dz \, d\bar{z}$. For $q = (q_1, q_2) \in Q$ we have one-dimensional projectors $P_q = |\zeta\rangle\langle\zeta|$, where

$$|\zeta\rangle = \frac{1 - |\zeta|^2}{(1 - \bar{\zeta} z)^2} \qquad \text{and} \qquad \zeta = \frac{1 - q_2 + iq_1}{1 + q_2 - iq_1}$$

The semispectral measure $E(B)$, $B \in \mathscr{B}(Q)$ is given by $E(B) = \int_B P_q \, d\alpha(q)$, where $d\alpha$ is the unique $SL(2, \mathbf{R})$ invariant measure on Q normalized in such a way that $\int_Q P_q \, d\alpha(q) = I$, the identity operator. The quantum operator corresponding to a function f on Q reads

$$\hat{f} = \int_Q f(q) \, dE(q)$$

Let us assume that the quantum system evolves according to the Hamiltonian $H=\overline{d\pi_1(h)}$, where $h=\sigma_3$, the Pauli matrix. It implies that $\exp(-th)=\mathrm{diag}(e^{-t}, e^{t})$.

To define a generator L of the total system we assume that $\lambda(x, p)=\lambda$, is a constant function, and that $h_q\colon M\to M$ is given by

$$h_{q_1, q_2}(x_1, x_2, p_1, p_2)=(x_1, x_2, p_1, p_2-q_2)$$

Hence

$$B(\tilde{A})(x, p)=(\delta_c\otimes id)\,\tilde{A}(x, p)+(id\otimes\delta_q)\,\tilde{A}(x, p)$$
$$+\lambda\int_Q d\alpha(q)\,P_q\tilde{A}(h_q(x, p))\,P_q-\lambda\tilde{A}(x, p)$$

The corresponding Markov–Feller generator reads

$$(\mathscr{L}f)(x, p, q)=Yf(x, p, q)+\lambda\int_E \nu(x, p, q; dx', dp', dq')\,f(x', p', q')$$
$$-\lambda f(x, p, q)$$

where

$$Yf(x, p, q)=\frac{d}{dt}f(g_t(x, p), \exp(-th)\,q)|_{t=0}$$

and

$$\nu(x, p, q; dx', dp', dq')$$
$$=Tr(P_qP_{q'})\,\delta(x'-x)\,\delta(p_1'-p_1)\,\delta(p_2'-(p_2-q_2'))\,dx'\,dp'\,d\alpha(q')$$

Here $g_t(x, p)$ is a geodesic curve on Q starting at point $x\in Q$ and with the tangent vector $p\in\mathbf{R}^2$, and $\exp(-th)(q_1, q_2)=(e^{-2t}q_1, e^{-2t}q_2)$. To finish the description of the generator $\mathscr{L}$ we now compute $Tr(P_qP_{q'})$ and the measure $d\alpha$. Because $Tr(P_qP_{q'})=|\langle q, q'\rangle|^2$ so

$$Tr(P_qP_{q'})=\frac{(1-|z|^2)(1-|z'|^2)}{(1+|z|^2\,|z'|^2-\bar{z}z'-z\bar{z}')^2}$$

where

$$z=\frac{1-q_2+iq_1}{1+q_2-iq_1}\qquad\text{and}\qquad z'=\frac{1-q_2'+iq_1'}{1+q_2'-iq_1'}$$

The measure $d\alpha$ is $SL(2, \mathbf{R})$ invariant and normalized in such a way that $\int_Q d\alpha(q)\, P_q = \mathbf{1}$, so $d\alpha(q) = (1/4\pi q_2^2)\, dq_1\, dq_2$. It implies that

$$\int_E v(x, p, q; dx', dp', dq')\, f(x', p', q')$$

$$= \int_E d\alpha(q')\, Tr(P_q P_{q'})\, f(x, p_1, p_2 - q_2', q')$$

A particular realization of this process can be described as follows. Let us assume that at time $t = 0$ the classical particle starts in point $(0, 1) \in Q$ with velocity $(1, 0)$, while the quantum particle is in the state $|q(0)\rangle = |(0, 1)\rangle$. Under the time evolution the classical particles moves along a curve $\gamma(t) = (\tanh t, \cosh^{-1} t)$, that is on the circle with the center in point $(0, 0)$ and with radius one. The quantum system evolves to the state $|q(t)\rangle$, where $q(t) = (0, e^{-2t})$. Then at time t_1 a jump occurs. The time rate of jumps is governed by a homogeneous Poisson process with rate λ. The quantum system jumps to a new state $|q'\rangle$ with probability $Tr(P_{q(t_1)} P_{q'})\, d\alpha(q')$, while the classical particle changes its velocity

$$\left(\cosh^{-2} t_1, -\frac{\sinh t_1}{\cosh^2 t_1}\right) \to \left(\cosh^{-2} t_1, -\frac{\sinh t_1}{\cosh^2 t_1} - q_2'\right)$$

$q_2' \in (0, \infty)$, with probability $p(q_2(t_1), q_2')$ given by

$$p(q_2(t_1), q_2') = \frac{1}{4\pi} \int_{-\infty}^{\infty} Tr(P_{q(t_1)} P_{q'})\, \frac{dq_1'}{(q_2')^2}$$

Because

$$Tr(P_{q(t_1)} P_{q'}) = \frac{16 q_2^2(t_1) \cdot 16 (q_2')^2}{(1 + q_2(t_1))^4}$$

$$\cdot \left[(1 + q_2')^2 + (q_1')^2 + \left(\frac{1 - q_2(t_1)}{1 + q_2(t_1)}\right)^2 ((1 - q_2')^2 + (q_1')^2) \right.$$

$$\left. - 2 \left(\frac{1 - q_2(t_1)}{1 + q_2(t_1)}\right) (1 - (q_1')^2 - (q_2')^2) \right]^{-2}$$

so

$$p(q_2(t_1), q_2') = 2 q_2^2(t_1) [q_2' + q_2(t_1)]^{-3}$$

And the process starts again. The classical particle moves again along a geodesic curve, but a different one, which is the circle with the center in point $(-q'_2 \cosh t_1, 0)$ and with radius

$$r = [1 + 2q'_2 \sinh t_1 + (q'_2 \cosh t_1)^2]^{1/2}$$

In other words the x_1-axis acts as an attracting boundary for the classical particle. By this we mean that the particle tries to reach this boundary faster than in the free case, and that the dynamical curves are "more curved" then the geodesic ones.

5. CONCLUDING REMARKS

The presented framework starts with a phenomenological assumption. At the very beginning we divide the world into two parts: a classical and quantum one, which are assumed to interact. The coupling induces a flow of information from the quantum system to the classical system. Such a coupling is obtained by means of a dynamical semigroup of the total system. To a given semigroup we associate the Markov–Feller process. In this process the randomness appears as point events, i.e., there is a sequence of random occurrences at random times, but there is no additional component of uncertainty between jump times. It consists of the mixture of deterministic motion with random jumps. The motion between jumps is determined by a complete vector field, while the jump mechanism by a Poisson measure. As was shown on the example it leads to an essential modification of the classical trajectories.

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